Astro 201; Project Set #2

due friday 2/24/2012

Problems on radiative diffusion in spherically symmetric media and on radiative equilibrium and radiation hydrodynamics for simple systems.

1. Obscured Active Galactic Nuclei

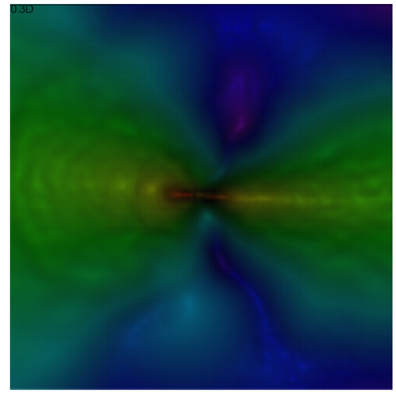

Figure 1: A snapshot of a 3-D hydrodynamical simulation of a dusty torus of gas formed around an AGN. The scale of the image box is $\sim$ 10 parsecs and the BH is an unresolved point at the center. Model from Hopkins and Quataert (2010) ; image rendered by Nathan Roth.

Accretion onto a supermassive black hole (BH) at the center of a galaxy can power strong radiation – an *active galactic nucleus* (AGN). The BHs powering AGN have masses $\sim 10^7 - 10^8\ M_\odot$ and should radiate mainly in the UV/x-ray. However, in some cases (e.g., the Seyfert 2 AGN) strong emission is seen at around $\sim 10\ \mu$m. It is thought that this infrared radiation is due to the absorption and remission of radiation in dusty gas surrounding the BH. The dusty region is thought to have a spatial extant of $\sim 1 - 10$ pc and a shape like a torus (see figure 1). The origin and properties of AGN dusty torii are an active area of research, with many efforts to model the observed IR spectra with radiation transport codes.

In this project, we develop a simple model of obscured AGN. Although the real systems are clearly aspherical, we'll solve the symmetric analogue – a spherical source surrounded by a spherical envelope.[1]. We first calculate the temperature of dusty gas being heated by a central BH. Once the temperature of the gas is known, we can calculate the spectrum of its infrared emission.

[1] Our simple setup may be applicable in other astrophysical contexts, e.g., a starburst occurring in a dusty galaxy, a massive star forming in a dusty cloud, or a supernova exploding inside a dusty circumstellar region. One would just need to change the length and luminosity scale.

We'll assume that BH has mass $M_{\rm BH} \sim 10^7\ M_\odot$ and emits a luminosity, $L_{\rm BH}$, equal to it's (electron-scattering) Eddington luminosity. We'll model the source of BH radiation as an isotropically emitting sphere[2] of radius $R_{\rm in} \sim 10$ Swarchzhild radii ($\sim$ 2 AU). Surrounding the source is a spherical envelope of mainly hydrogen gas with some dust mixed in. We'll take the envelope to have a constant density ρ_0 extending from $R_{\rm in}$ to an outer radius $R_{\rm out} \sim 10$ parsecs (so $R_{\rm out} \gg R_{\rm in}$). We'll assume that radiative heating/cooling dominates the energy exchange in the envelope, so that (given enough time) the envelope will come into radiative equilibrium.

[2] Of course, the emission is actually coming from the accreting material just outside the BH, which is presumably disk-like (not spherical) and does not emit isotropically. But anyway.

The optical depth of the envelope (measured radially, from the center to the edge) is $\tau_0 \simeq \rho_0 \kappa R_{\rm out}$, where $\kappa \sim 10$ cm^2 g^{-1} is a typical infrared opacity of dusty gas[3], which we will take to be purely absorptive. We'll solve the radiation transport problem separately for the two limits: $\tau_0 \ll 1$ (optically thin) and $\tau_0 \gg 1$ (optically thick). We won't attempt the intermediate case ($\tau_0 \sim 1$) which is actually the hardest to solve, since no simplifying approximation to the transport can be made.

[3] The gas and the dust are typically tightly collisionally coupled (i.e., quickly come into thermal equilibrium with each other) and so we treat them as a single fluid. By opacity, we thus mean the cross-section per unit gram of dusty gas. Most of the mass of this fluid is from hydrogen, so the cross-section of this material is $\sigma \simeq m_p \kappa_{\rm IR}$ where m_p is the proton mass.

In general, this is a time-dependent problem – i.e., the temperature and density of the envelope is evolving under the influence of gravity[4] and radiation feedback. Here we'll make the *stationarity approximation* – we take a snapshot in time in which the envelope structure is held fixed and solve the steady state radiation transport problem. We'll check the validity of the assumption as we go.

[4] You can safely neglect the self-gravity of the envelope; the gravity of the BH dominates.

I. Optically thin case

In the optically thin limit, most photons free-stream through the envelope without interacting, and the radiation field can be approximated by the value it would have in empty space[5]. We take the opacity to be independent of wavelength.

[5] It is true that a fraction $\sim \tau_0$ of the photons are absorbed in the envelope, and so the radiation field is not exactly the same as it would be in empty space, but let's not worry about that small fraction.

a) Assume that the specific intensity from the surface of our spherical source is given by Planck's function at a temperature T_s. What is T_s? At around what wavelengths does the BH radiate?

b) Write down the expression for the mean intensity, $J(r)$, at all radii outside the source in this optically thin case. Use it to solve for the temperature profile, $T(r)$ (in terms of T_s and R_{in}) of the dusty envelope assuming that it is in radiative equilibrium. What is a characteristic temperature of the envelope?[6] At around what wavelengths does the envelope radiate?

[6] Most of the mass of the envelope is at large radii, so you might evaluate $T(r)$ at, say, $r \sim R_{out}/2$.

c) Let's check that our use of the stationarity approximation is reasonable. What is the characteristic time scale, t_{esc} for photons to escape the dusty envelope? How does this compare to the dynamical timescale, t_{dyn} (e.g., the time it would take to outer edge of the envelope to free-fall into the black hole)? Also check that the timescale, t_{eq}, for the envelope to come into radiative equilibrium is short compared to the dynamical timescale.[7] Does it seem safe to assume the envelope structure is fixed when solving this radiation transport problem?

[7] As a rough test, it will suffice to look at the value of either the cooling or heating time for a characteristic equilibrium temperature at, say, $R_{out}/2$.

II. Optically thick case

Next consider the opposite limit, in which the envelope is optically thick and the diffusion approximation applies. Continue to assume the opacity is grey and purely absorptive. The diffusion equation in spherical coordinates is

$$L(r) = -4\pi r^2 \frac{c}{3\kappa\rho} \frac{\partial}{\partial r} u(r) \tag{1}$$

Assume that radiative equilibrium holds, in which case the luminosity is constant with radius (and equal everywhere to $L_{\rm BH}$).

d) What is the expression for the characteristic time scale, $t_{\rm esc}$ for photons to escape the dusty envelope in this optically thick case? At about what value of τ_0 does our stationarity approximation become questionable?

e) Solve the diffusion equation to determine the temperature profile of the dusty envelope $T(r)$ in this optically thick case. You'll need to specify a boundary condition – we'll take it to be that the temperature is zero at $R_{\rm rout}$, the so-called "radiative zero" boundary condition.[8]

[8] This is not necessarily the best boundary condition. A better one might be that $T(R_{\rm out}) = T_{\rm eff}$, where we define the effective temperature by $L_{\rm BH} = 4\pi R_{\rm out}^2 \sigma_{\rm sb} T_{\rm eff}^4$.

Bonus (optional): Consider the thin surface layer of the envelope just below $R_{\rm out}$, for which the plane parallel approximation applies. Show that the temperature structure of this "atmosphere" follows $T^4(\tau_z) \propto \tau_z$, where the optical depth coordinate, τ_z, is defined to be zero at the $R_{\rm out}$ and increases inwards. You'll notice that this is the same sort of temperature dependence you derived by solving the radiative transport equation for a plane-parallel stellar atmosphere in problem 2e) of project set #1. [9]

[9] To get the exact same relation, $T^4(\tau_z) \propto (\tau_z + 2/3)$, you'll have to choose a smarter boundary condition for the diffusion equation. The correspondence here is not totally surprising, as in the previous problem we assumed that the angular dependence of the radiation field was linear in μ, which is the same sort of assumption that goes into the diffusion approximation.

Bonus (optional): Our assumption of a constant density envelope is a little unrealistic. However, you can easily solve the same diffusion problem using a power-law density profile $\rho(r) = \rho_0 (r/r_{\rm in})^{-\zeta}$.

f) Plot the two temperature profiles you have derived (optically thin and optically thick cases) in comparison to each other. For the optically thick case, take $\tau_0 = 100$. For the optically thin case take $\tau_0 = 0.1$. Note that dust is sublimated (destroyed) at temperatures higher than $\sim$ 1500 K. Within about what radii do expect dust destruction to be important?

g): Consider the dusty gas at about the middle of the envelope ($r = R_{\rm out}/2$). How does the temperature you find for the optically thick case compare to that of the optically thin case? Argue why the ratio of these temperatures makes physical sense.

III. The emergent spectrum

Now that we have calculated the temperature structure of the envelope (in the limiting cases at least), we can model the emergent spectrum and compare to observations.

h) Numerical: For the optically thin case, write a simple code to integrate the wavelength dependent emissivity over the entire volume[10]

[10] If you are not into programing, you can find my simple python code on the class website . At the least look it over to see how this can be done.

and so calculate the infrared spectrum of the envelope (ignore the UV emission from the BH). For the optically thick case, we can just approximate the spectrum by a blackbody at the temperature T_{eff}. Plot up the two spectra and compare.

Comment: Our model is too simple to apply to real obscured AGN, for many reasons: (1) We have assumed a spherical geometry, when in reality the envelope is probably a torus; (2) We have used a grey opacity, when in reality the dust opacity varies with wavelength; (3) We have assumed the density distribution is smooth and uniform, while more detailed models suggest that it is highly clumpy. Nevertheless, our analytic solutions may be useful for testing full blown radiation transfer codes. They may also provide some intuition into the results of detailed 3-D calculations (e.g., the optically thin solution might best correspond to a more polar view, where the gas column densities are lower, while the optically thick solution might better correspond to an equatorial view).

2. *A Solar Supernova*

When disaster movies were all the rage several years ago, a few (bad) scripts floated around about the sun going supernova. None made it to the big screen. That does not mean that we should consider ourselves safe. Therefore in this project we'll develop an analytic model of what such a catastrophe would look like. The formalism we derive is actually extremely useful in analyzing the light curves of real supernovae and other astrophysical transients[11].

In a real supernova, an energy of $E \sim 10^{51}$ ergs $= 1B$ is deposited (somehow) in the core of a star. This generates a shockwave of characteristic velocity of $v \sim \sqrt{2E/M} \sim 10,000$ km s^{-1} which propagates through the star, depositing about half of the energy in the form of internal energy and the other half as kinetic energy of the ejected debris[12]. Because it takes only a short time ($t_0 = R_\odot/v \sim 1$ min) for the shockwave to traverse a sun-like star, we very rarely see the shock emergence itself – what we do typically observe in the days and weeks to follow is the thermal radiation diffusing out of the opaque, expanding remnant.

We'll approximate the remnant as a uniform density, isothermal, expanding sphere of mass M, and initial radius R_0. The remnant is a hot plasma of ionized hydrogen, so the opacity is dominated (at most wavelengths at least) by electron scattering.

a) What is (roughly) the optical depth (radially from the center to the

[11] We follow a simple version of the more detailed analytic model for supernova light curves derived in a classic paper, Arnett 1980 .

[12] This nearly equal split of the energy between internal and kinetic in the post-shock region follows from the hydrodynamics of strong (i.e., highly supersonic) shocks.

surface) of the remnant right after the sun has exploded (i.e., when the remnant radius is still $R_0 = R_\odot$)?

b) Given the high optical depth, the radiation is (at least initially) trapped in the remnant. Let's assume that the gas and radiation are well thermally coupled and that both have equilibrium distributions of the same temperature. What is the characteristic temperature, T_0, of the remnant right after the sun has exploded with an energy $E = 1$ B? Show that the radiation energy density exceeds the gas energy density by quite a margin[13]. A supernova is basically a big fireball of radiation, held in by the opacity of the gas.

[13] It's easiest to do this in reverse – first calculate T_0 assuming that radiation energy dominates, then verify that your assumption is indeed true.

Shortly after a the explosion, gravity and pressure forces become negligible and the remnant reaches a phase of free expansion. The radius of the remnant is then given by $R(t) \simeq vt$, where $v \approx \sqrt{2E/M}$ is the characteristic expansion velocity, R_0 is the initial radius, and t is time. In this case (known as *homologous* expansion) the volume of the remnant increases with time as $V(t) = V_0(t/t_0)^3$, where the initial volume is $V_0 = (4\pi/3)R_0^3$ and $t_0 = R_0/v$ is the time it took for the shock to explode the star.

As the remnant progressively expands, the internal energy of the remnant will evolve according to the energy equation (i.e., the first law of thermodynamics)

$$\frac{dE}{dt} = -p\frac{dV}{dt} - L(t) \tag{2}$$

where $E(t) = u(t) \times V(t)$ is the internal energy and $p(t)$ the pressure of the remnant (both dominated by radiation). $L(t)$ is the luminosity (ergs s^{-1}) escaping the remnant (i.e., the supernova light curve) which is what we want to figure out.

c) First, to orient ourselves, consider the case of adiabatic expansion where $L = 0$ (i.e., no heat is entering or leaving the system). This is appropriate for the earliest times ($\lesssim 1$ day) when the remnant is still very opaque and very little radiation escapes. Show that temperature of a homologous expanding, radiation dominated remnant cools adiabatically as $T(t) = T_0(t/t_0)^{-1}$. In this adiabatic approximation, about how hot do we expect the debris from the solar supernova to be when it rains down on the earth[14]?

[14] ouch.

To solve the non-adiabiatic case, let's approximate the escaping luminosity $L(t)$ by the diffusion equation in spherical coordinates (equation 1). To fully solve this equation we would need to specify the radial structure of the remnant, i.e., $\rho(r), \epsilon(r)$. However, all we really want is an order of magnitude estimate of $L(t)$. In such cases, a useful trick for approximating the value of a spatial derivative is to just to take a "one-zone" value and assume the quantity changes by its full value over the characteristic length scale

$$\frac{\partial u}{\partial r} \approx -\frac{u(t)}{R(t)} \tag{3}$$

d) Using this one-zone approximation, solve equation 2 for the energy density $u(t)$ as a function of time. Then use the diffusion equation[15] to derive an analytic formula for the supernova light curve, $L(t)$. What are simple expressions (in terms of the basic physical parameters E, R_0, κ, and M) for the characteristic luminosity, $L_{\rm sn}$, of the supernova and the characteristic time scale, $t_{\rm sn}$, on which the supernova luminosity declines?

[15] You can continue to approximate the spatial derivative by equation 3. Note that the radius and density in this equation also evolve with time. You can replace them with $r(t) = vt$ and $\rho(t) = M/V(t)$

Comment: The timescale $t_{\rm sn}$ you have derived gives the effective diffusion time in a (homologously) expanding medium, which is in general useful for optically thick outflows. It is different than the familiar static diffusion time because the density (and hence optical depth) drop as the remnant expands, making it easier for photons to escape. Note that the diffusion time in a static medium can be written $t_{\rm d} \sim \kappa M/R_0 c$. You have therefore shown that $t_{\rm sn} \propto \sqrt{t_{\rm d} t_{\rm e}}$, where $t_{\rm e} \approx R_0/v$ is the the characteristic expansion time of the supernova remnant. In other words, the time it takes photons to diffuse out of an expanding medium is given by the geometric mean of $t_{\rm d}$ and $t_{\rm e}$. For the sun, $t_d \sim 10^4$ years, whereas $t_e \approx 1$ minute.

e) Now we will blow up the sun. Dump an energy of $E = 1$ B into our life-giving star and plot up your predicted light curve. Roughly how bright (in solar luminosities) is our solar supernova and about how long does it last?[16]

[16] Of course, you will notice that the remnant will overrun the earth long before most of the radiation has had a chance to escape. Thus, unfortunately, we will never really be able to appreciate the beauty of our solar supernova.

f) Using a rough estimate of the total energy radiated by the supernova ($E_{\rm sn} \sim L_{\rm sn} t_{\rm sn}$) determine the percentage of the initial internal energy of the remnant that winds up escaping as radiation. Where does the rest of the internal energy go?

g) Assuming the luminosity is radiated approximately as a blackbody, at around what wavelengths does our solar supernova shine?

h) The most common type of supernovae (the Type IIP core collapse events) have luminosities of around 10^{42} ergs s^{-1}. How does this compare to your predicted value?

Comment This discrepancy caused some confusion when the first attempts to model Type II supernovae were made. The mystery was solved when people realized that the progenitors were not stars of solar radii, but in fact red supergiants, with radii ~ 100 larger than the sun. From your equation, you will note that this predicts a brighter light curve in rough agreement with observations.

Comment: Type Ia supernovae come from the thermonuclear explosion of white dwarfs, which have much smaller radii then the sun. And yet they are even brighter than most core collapse supernovae. The reason is that the remnant is continually being heated by the decay of radioactive isotopes produced in the explosion. In this case, the first law of thermodynamics is

$$\frac{dE}{dt} = -p\frac{dV}{dt} - L(t) + \dot{E}_{\rm rad} \tag{4}$$

where $\dot{E}_{\rm rad}$ is the energy per unit time being deposited by radioactivity, which follows a roughly exponential decline in time. There is no simple analytic expression for the light curve in this case, although the equation is easily solved numerically. You can however (**Bonus**) analytically solve the problem for the simplified scenario where $E_{\rm rad} = $ constant for $t < t_{\rm rad}$ and zero afterwards, where $t_{\rm rad} \sim 10$ days is a typical radioactive decay time.

Bonus (optional) – Astro 201: The Real World

We Must Blow Up the Moon

While doing the last problem, you might have wondered: "why would the sun ever go supernova?" Actually, there is no reason to think it ever will. There is some cause for concern, though, that the moon might blow up, perhaps even in our lifetime. If you don't believe me, watch this report .

These visionaries (Bob and Dave) report that the spacecraft Exploder 1 will "carry enough dynamite and nuclear charges to blow up the moon 50 times over." Given this information, estimate how bright and long lasting the "lunar supernova" will be and advise Bob and Dave on what it would actually look like from earth (assuming america ever gets it together to pull this off).